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There are investigated the problems of the flow of a supersonic jet out of a vessel with plane side walls and the problem of the supersonic flow about a wedge when there is a zone of local subsonic velocities ahead of the wedge.

INTRODUCTION

In the present paper it is assumed that the reader is acquainted with the work of S. A. Chaplygin ("On Gas Jets" (reference 1)) and with the method of computation of plane-parallel supersonic flows given by Prandtl and Busemann (reference 2, see also references 5 and 6). There is recommended a preliminary acquaintance with the work of F. Tricomi "On second order partial differential equations of mixed type" (reference 3) whose methods undoubtedly will be capable of being used in proving the existence of the solution of the problems considered by us.

Since in what follows we shall everywhere make use of the notation of Chaplygin we shall here present the formulas and notation of importance to use. Chaplygin makes use of the method of the hodograph. As the independent variables he chooses in the first place the nagnitude

$$\tau = \frac{V^2}{V^2_m} \tag{1}$$

where V is the flow velocity at a given point, V_m is the maximum velocity corresponding to the stagnation temperature T_0 (that is, the temperature arising in front of an obstacle in the flow) characteristic for the given flow; V_m is given by

$$V^2_m = 2Jgc_pT_0 \tag{2}$$

where J is Joule's constant, g the acceleration of gravity, c_p the specific heat for constant pressure, and T_0 the absolute

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stagnation temperature. The second independent variable is taken to be the angle of inclination of the velocity θ. Written in these independent variables the stream function ψ in the case of irrotational flow satisfies the equation

$$\frac{\partial}{\partial\tau}\left\{\frac{2\tau}{(1-\tau)^{\beta}}\frac{\partial\psi}{\partial\tau}\right\} + \frac{1-(2\beta+1)\tau}{2\ (1-\tau)^{\beta+1}}\frac{\partial^2\psi}{\partial\theta^2} = 0 \qquad (3)$$

where

$$\beta = \frac{1}{\kappa - 1} \qquad (3a)$$

and

$$\kappa = \frac{c_p}{c_v} \qquad (3b)$$

the ratio of specific heats at constant pressure and constant volume, respectively.

The value

$$\tau = \frac{1}{2\beta + 1} \qquad (4)$$

corresponds to the critical velocity; that is, the velocity of the flow equal to the corresponding local sound velocity.

On introducing the auxiliary variable

$$\sigma = \int_{\tau = \frac{1}{2\beta+1}}^{(2\beta+1)^{-1}} \frac{(1-\tau)^{\beta}}{2\tau}\, d\tau \qquad (5)$$

equation (3) assumes the form

$$\frac{\partial^2\psi}{\partial\sigma^2} + K\frac{\partial^2\psi}{\partial\theta^2} = 0 \qquad (6)$$

where

$$K = \frac{1-(2\beta+1)\tau}{(1-\tau)^{2\beta+1}} \qquad (7)$$

Thus, equation (6) for subsonic velocities will be of the elliptic type and for supersonic velocities of the hyperbolic type.

Chaplygin further considers particular solutions of equation (3) of the form

$$\psi_\nu (\tau, \theta) = z_\nu (\tau) \sin 2\nu\theta \tag{8}$$

where

$$z_\nu (\tau) = \tau^\nu y_\nu (\tau) \tag{9}$$

and $y_\nu (\tau)$ is the hypergeometric function

$$y_\nu (\tau) = F (a_\nu, b_\nu; 2\nu + 1; \tau) \tag{10}$$

where

$$a_\nu + b_\nu = 2\nu - \beta, \quad a_\nu b_\nu = - \beta\nu (2\nu + 1) \tag{10a}$$

In the theory of Chaplygin an important part is also played by the auxiliary function $x_\nu (\tau)$

$$x_\nu = 1 + \frac{\tau}{\nu} \frac{y'_\nu}{y_\nu} = \frac{\tau}{\nu} \frac{z'_\nu}{z_\nu} \tag{11}$$

The problems considered by Chaplygin for flow velocities remaining everywhere below the velocity of sound reduce to the problem of Dirichlet and are solved with the aid of series combined from the special solutions of the form (8). To what boundary problems for equation (3) the problems of Chaplygin reduce for mixed sub- and supersonic flows remained unknown. Basing himself on the work of Tricomi (reference 3), the author has succeeded in finding a formulation of these problems and to establish the uniqueness of their solutions.

In what follows the author hopes to give a mathematically well founded and practically suitable solution of the problems stated.

I. REDUCTION OF THE PROBLEMS OF THE FLOW OF A SUPERSONIC JET TO THE PROBLEM OF TRICOMI FOP THE EQUATION OF CHAPLYGIN UNIQUENESS THEOREM FOR THESE PROBLEMS

The problem of Tricomi is the following: Let there by given a linear partial differential equation of the second order which on one side of the curve C in the plane of the independent variables is of the elliptic type and on the other side is of the hyperbolic type. Let us consider the region D bounded by the curve L lying in the elliptic region with its ends lying on the curve C and with the characteristics x_1 and x_2 belonging to different families and starting from the ends of the curve L (fig. 1), Let the values of the solution be given on the curves L and x_1 but not on x_2. There is sought a solution in the region D.

This boundary problem was first formulated by F. Tricomi (reference 3) as applied to the equation

$$y \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \tag{1}$$

Tricomi proved the uniqueness and existence of the solution of this problem. In this section we shall reduce the problem of the flow of a supersonic jet to a certain problem of Tricomi for the equation of Chaplygin (see introduction equation (6)):

$$\frac{\partial^2 \psi}{\partial \sigma^2} + K \frac{\partial^2 \psi}{\partial \theta^2} = 0 \tag{2}$$

The coefficient K for small τ or large σ is equal to unity and drops with increasing τ (decreasing σ). For $\tau = (1 + 2\beta)^{-1}$ (or $\sigma = 0$) we have $K = 0$, and for $\tau > (1 + 2\beta)^{-1}$ (or $\sigma < 0$) $K < 0$. Thus equation (2) is of the elliptic type for $\sigma > 0$ and hyperbolic for $\sigma < 0$. We shall first prove the uniqueness of the solution of the problem of Tricomi for equation (2).

We consider in the plane (θ, σ) a finite region bounded by the curve ABC lying in the half plane $\sigma > 0$ and the characteristics AD, CD, lying in the half plane $\sigma < 0$ (fig. 2). We assume that in this region the solution ψ of equation (2) is taken equal to zero on ABC and on CD. We shall show that this solution is equal to zero over the entire region.

We consider first the solution in the triangle and will show that

$$\int_0^{\theta_0} \psi \frac{\partial \psi}{\partial \sigma} \, d\theta \Big|_{\sigma=0} \geqslant 0 \tag{3}$$

We transform the equation

$$\iint_{ADC} \psi \left(\frac{\partial^2 \psi}{\partial \sigma^2} - |K| \frac{\partial^2 \psi}{\partial \theta^2} \right) d\sigma \, d\theta = 0 \tag{4}$$

by integrating by parts. We have

$$\left. \begin{aligned} \int_0^{\sigma} \psi \frac{\partial^2 \psi}{\partial \sigma^2} \, d\sigma &= \psi \frac{\partial \psi}{\partial \sigma} \Big|_0^{\sigma_1} - \int_0^{\sigma_1} \left(\frac{\partial \psi}{\partial \sigma} \right)^2 d\sigma \\ \int_{\theta_1}^{\theta_2} \psi \frac{\partial^2 \psi}{\partial \theta^2} \, d\theta &= \psi \frac{\partial \psi}{\partial \theta} \Big|_{\theta_1}^{\theta_2} - \int_{\theta_1}^{\theta_2} \left(\frac{\partial \psi}{\partial \theta} \right)^2 d\theta \end{aligned} \right\} \tag{5}$$

Hence

$$0 = \iint_{ADC} \psi \left(\frac{\partial^2 \psi}{\partial \sigma^2} - |K| \frac{\partial^2 \psi}{\partial \theta^2} \right) d\theta \, d\sigma = \int_{DA} \psi \left(- \frac{\partial \psi}{\partial \sigma} \, d\theta - |K| \frac{\partial \psi}{\partial \theta} \, d\sigma \right) -$$

$$- \iint_{ADC} \left[\left(\frac{\partial \psi}{\partial \sigma} \right)^2 - |K| \left(\frac{\partial \psi}{\partial \theta} \right)^2 \right] d\theta \, d\sigma + \int_0^{\theta_0} \psi \frac{\partial \psi}{\partial \sigma} \, d\theta \Big|_{\sigma=0} \tag{6}$$

Along the characteristic DA we have

$$\left. \begin{aligned} d\theta - \sqrt{|K|} \, d\sigma &= 0 \\ d\sigma = \frac{d\theta}{\sqrt{|K|}}, \quad d\theta &= \sqrt{|K|} \, d\sigma \end{aligned} \right\} \tag{7}$$

Hence

$$\int_{DA} \psi\left(\frac{\partial \psi}{\partial \sigma}\, d\theta + |K|\, \frac{\partial \psi}{\partial \theta}\, d\sigma\right) = \int_{DA} \sqrt{|K|}\, \psi \left(\frac{\partial \psi}{\partial \sigma}\, d\sigma + \frac{\partial \psi}{\partial \theta}\, d\theta\right) =$$

$$= \int \sqrt{|K|}\, \psi\, d\psi = \underbrace{\frac{\psi^2}{2}\sqrt{|K|}\Big|_{\sigma_{min}}^{0}}_{=0} - \frac{1}{2}\int_{DA} \frac{d\sqrt{|K|}}{d\sigma}\, \psi^2\, d\sigma \tag{8}$$

so that

$$\int_0^{\theta_0} \psi\, \frac{\partial \psi}{\partial \sigma}\Big|_{\sigma=0}\, d\theta = -\frac{1}{2}\int_{DA} \frac{d\sqrt{|K|}}{d\sigma}\, \psi^2\, d\sigma + \iint_{ADC} \left[\left(\frac{\partial \psi}{\partial \sigma}\right)^2 - |K|\left(\frac{\partial \psi}{\partial \theta}\right)^2\right] d\theta\, d\sigma \tag{9}$$

We now compute $\frac{d\sqrt{|K|}}{d\sigma}$:

$$\frac{d\sqrt{|K|}}{d\sigma} = \frac{1}{2\sqrt{|K|}}\, \frac{d\,|K|}{d\tau}\left(-\frac{2\tau}{(1-\tau)^{\beta}}\right) = -\frac{\tau}{(1-\tau)^{\beta}\sqrt{|K|}}\, \frac{2\beta\,(2\beta+1)\,\tau}{(1-\tau)^{2\beta+2}} < 0 \tag{10}$$

Thus, to prove the inequality (3) it remains to show that

$$\iint_{ADC} \left[\left(\frac{\partial \psi}{\partial \sigma}\right)^2 - |K|\left(\frac{\partial \psi}{\partial \theta}\right)^2\right] d\theta\, d\sigma \geq 0 \tag{11}$$

To prove this we shall transform to characteristic coordinates:

$$\left.\begin{aligned} d\lambda &= d\theta + \sqrt{|K|}\, d\sigma \\ d\mu &= d\theta - \sqrt{|K|}\, d\sigma \end{aligned}\right\} \tag{12}$$

from which we obtain the Jacobian determinant:

$$\frac{D\,(\lambda,\ \mu)}{D\,(\theta,\ \sigma)} = -\,2\sqrt{|K|} \tag{13}$$

Differential equation (3) becomes

$$-\,4\,|K|\,\frac{\partial^2\psi}{\partial\lambda\,\partial\mu} + \frac{d\sqrt{|K|}}{d\sigma}\left(\frac{\partial\psi}{\partial\lambda} - \frac{\partial\psi}{\partial\mu}\right) = 0 \tag{14}$$

On the other hand

$$\left(\frac{\partial\psi}{\partial\sigma}\right)^2 - |K|\left(\frac{\partial\psi}{\partial\theta}\right)^2 = -\,4\,|K|\frac{\partial\psi}{\partial\lambda}\,\frac{\partial\psi}{\partial\mu} \tag{15}$$

or

$$\iint\left[\left(\frac{\partial\psi}{\partial\sigma}\right)^2 - |K|\left(\frac{\partial\psi}{\partial\theta}\right)^2\right] d\theta\ d\sigma = -\,2\iint\sqrt{|K|}\,\frac{\partial\psi}{\partial\lambda}\,\frac{\partial\psi}{\partial\mu}\ d\lambda\ d\mu \tag{15a}$$

where the integral on the right is taken over the area C'A'D' (fig. 3)

To compute this integral we rewrite equation (14) in the following form:

$$\sqrt{|K|}\left(\frac{\partial\psi}{\partial\lambda} - \frac{\partial\psi}{\partial\mu}\right) = M\ (\sigma)\ \frac{\partial^2\psi}{\partial\lambda\,\partial\mu} \tag{16}$$

where

$$M\ (\sigma) = \frac{4\,|K|^{3/2}}{\dfrac{d\sqrt{|K|}}{d\sigma}} \tag{16a}$$

Then

$$\iint\sqrt{|K|}\,\frac{\partial\psi}{\partial\lambda}\,\frac{\partial\psi}{\partial\mu}\ d\lambda\ d\mu = \iint\sqrt{|K|}\left(\frac{\partial\psi}{\partial\mu}\right)^2 d\lambda\ d\mu + \iint M\,\frac{\partial\psi}{\partial\mu}\,\frac{\partial^2\psi}{\partial\lambda\,\partial\mu}\ d\lambda\ d\mu =$$

$$= \iint\left(\sqrt{|K|} - \frac{1}{4\sqrt{|K|}}\,\frac{dM}{d\sigma}\right)\left(\frac{\partial\psi}{\partial\mu}\right)^2 d\lambda\ d\mu + \frac{1}{2}\int M\left(\frac{\partial\psi}{\partial\mu}\right)^2\Bigg|_{\lambda=0}^{\lambda=\mu} d\mu =$$

$$= \iint\left(\sqrt{|K|} - \frac{1}{4\sqrt{|K|}}\,\frac{dM}{d\sigma}\right)\left(\frac{\partial\psi}{\partial\mu}\right)^2 d\lambda\ d\mu \tag{17}$$

since for small σ

$$M = 0\ (\sigma^2) \tag{17a}$$

and for continuous $\partial\psi/\partial\theta$, $\partial\psi/\partial\sigma$

$$\frac{\partial\psi}{\partial\mu} = 0\ (|\sigma|^{-1/2}) \tag{17b}$$

On the other hand,

$$\sqrt{|K|} - \frac{1}{4\sqrt{|K|}} \frac{dM}{d\sigma} = \sqrt{|K|}\, \frac{(2 + \beta)\ \tau - 2}{\beta\ (2\beta + 1)\ \tau^2} \tag{18}$$

This expression is negative if over the entire triangle ACD

$$\tau < \frac{2}{2 + \beta} \tag{19}$$

The above inequality expresses the fact that the Mach number M should be less than 2; for, with $\tau = \dfrac{2}{2 + \beta}$

$$M^2 = \frac{2}{\kappa - 1}\, \frac{\tau}{1 - \tau} = \frac{2}{\kappa - 1}\, \frac{2\ (\kappa - 1)}{2\kappa - 1}\ (2\kappa - 1) = 4 \tag{19a}$$

This means for $\kappa = 1.4$ the base θ_0 of the triangle must satisfy the inequality

$$\theta_0 < \bar{\theta} \cong 54^{\circ} \tag{19b}$$

Whether this restricting condition for the proof of the inequality (3) is essentially required or whether it is only connected with our method of proof is as yet unclarified*.

Let us now consider the region ABC. By integrating by parts we obtain as above:

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*The proof remains valid for any equation of the form (2) where K is a regular function of σ for $\sigma = 0$ and $dK/d\sigma > 0$ for $\sigma = 0$ and $K(0) = 0$, provided that σ in the triangle ADC remains sufficiently small. As applied to equation (1) the proof remains valid for any size triangles ADC. The same is true of the proof of the uniqueness as a whole.

$$0 = \iint_{ABC} \psi \left(\frac{\partial^2 \psi}{\partial \sigma^2} + K \frac{\partial^2 \psi}{\partial \theta^2} \right) d\theta \, d\sigma = - \int_0^{\theta_0} \psi \left. \frac{\partial \psi}{\partial \sigma} \right|_{\sigma=0} d\theta -$$

$$- \iint_{ABC} \left[\left(\frac{\partial \psi}{\partial \sigma} \right)^2 + K \left(\frac{\partial \psi}{\partial \theta} \right)^2 \right] d\theta \, d\sigma \tag{20}$$

whence

$$\int_0^{\theta_0} \psi \left. \frac{\partial \psi}{\partial \sigma} \right|_{\sigma=0} d\theta \leqslant 0 \tag{21}$$

and taking account of inequality (3)

$$\int_0^{\theta_0} \psi \left. \frac{\partial \psi}{\partial \sigma} \right|_{\sigma=0} d\theta = 0 \tag{22}$$

From equations (24), (22), (11), (17a), and (19) it follows now that

$$\psi = 0 \tag{23}$$

as was to be proved.

The proof of uniqueness here given is applicable under the condition that the transformations encountered are valid. This completion of the uniqueness proof we shall give after investigating the properties of the solution of our problem.

Let us return to the problem of the flow of a supersonic stream. We consider a vessel with symmetrically arranged walls forming an angle $2\theta^0$ (fig. 4(a)). We assert that for a sufficiently small external pressure the problem is reduced to the following problem of Tricomi: In the region OA'B'C (fig. 4) a solution ψ is sought under the conditions:

$$\left. \begin{aligned} &1^0 \quad \psi = -\frac{Q}{2} \text{ on } OA'B' \\ &2^0 \quad \psi = 0 \quad \text{ on } OC \end{aligned} \right\} \tag{24}$$

Similarly on the boundary of the region OABC we must have

$$\left.\begin{aligned} &1^\circ \quad \psi = \frac{Q}{2} \text{ on } OAB \\ &2^\circ \quad \psi = 0 \text{ on } OC \end{aligned}\right\} \tag{24a}$$

The obtained solution gives a mapping of the region OA'B'CBAO in the plane (x, y); thereby are obtained the region of subsonic velocities, the curve of sound velocities, and the part of the supersonic stream touching this curve. The continuation of the supersonic stream to infinity is obtained by the method of Prandtl and Busemann (reference 2).

We now proceed to prove the above statement. According to the conditions (24) and (24a) the walls of the vessel correspond to the radii OA' and OA, and the axis of symmetry of the vessel to the radius OC. As regards the characteristics AB and A'B' there correspond to them in the (x, y) plane the points A and A', the opening edges of the vessel. For, if along the characteristic of equation (4) the stream function ψ is constant, the potential φ is likewise constant (reference 4, section I, formulas 1.15 and 1.16). But if along a certain line φ and ψ are constant, then the coordinates x and y are also constant. It remains to show that the obtained flow may be continued in the form of a stream with constant pressure (constant velocity) on its boundaries. Such continuation is possible if the ratio of pressures p_1/p_0 is less than (not equal to) a certain function of the angle θ:

$$\frac{p_1}{p_0} \leqslant \frac{p_B}{p_0} = f\ (\theta_0) \quad (P_B - \text{pressure at point B}) \tag{25}$$

We give below a table of values of this function:

θ_0 =	10°	20°	30°	40°	50°	54°
$f\ (\theta_0)$ =	0,33	0,26	0,20	0,17	0,14	0,13

Let us consider first the case $p_1 = p_B$. We draw the arc of a circle BB' with center at the origin of coordinates, and prolong the characteristics A'B' and AB to their intersection D.

We now find the solution $\psi = \psi_2$ of the equation of Chaplygin in the triangle B'CB passing along the characteristic CB into the previous solution $\psi = \psi_1$ and equal to (+ Q/2) along BB'. Further, we find the solution $\psi = \psi_3$ in the triangle BB'C passing into the solution $\psi = \psi_1$ along CB' and equal to $(-Q/2)$ along B'B. Further, we find the solution $\psi = \psi_4$ in the quadrilateral CB'DB passing along CB into ψ_3 and along CB' into ψ_2. Continuing, we find the solution ψ_5 equal to ψ_4 along B'D and to $+Q/2$ along B'B, and symmetrical to the latter, the solution ψ_6 equal to ψ_4 along BD and equal to $(-Q/2)$ along BB'. We then find the solution ψ_7 equal to ψ_5 along DB and equal to ψ_6 along DB', etc.

The regions in the plane (x, y) corresponding to these solutions are denoted in figure 4 by the corresponding numbers. Thus we evidently obtain the flow with pressure p_B on the boundary.

If the pressure in the outer region is less than p_B we proceed as follows (fig. 5). We draw the arc of a circle EE' with radius corresponding to the pressure $p_1 < p_B$. The points E and E' must lie on the prolongations of the characteristics AB and A'B'. The intersections of this circle with the prolongations of the characteristics CB and CB' we denote by F and F'. We draw finally through F and F' the conjugates of the characteristics intersecting in the point D on the axis of u.

We now find the solution ψ_2 in the quadrilateral CBEF' that passes into ψ_1 along CB and is equal to $Q/2$ along BDF'; then the solution ψ_3 in the quadrilateral CB'EF that passes into ψ_1 along CB' and is equal to $(-Q/2)$ along B'E'F'. Further, we find the solution ψ_4 in the qualdrilateral CF'DF which passes into ψ_2 along CF' and into ψ_3 along CF. There are then found two solutions in the triangle FF'C, etc. as was shown in the previous case. We thus obtain the flow with the constant pressure $p_1 < p_B$ on the boundary.

To prove the existence of a steady continuous supersonic stream flowing out of a vessel it is necessary only to prove the existence of the solution of the problem of Tricomi*. A strict proof of existence, as has already been said in the introduction, has not yet been obtained by us. The fact, however, that the solution of the problem of Tricomi for equation (1) exists and for equation (2) there has been shown the uniqueness of the solution makes it appear probable that the solution of the problem of Tricomi for equation (2) likewise exists.

It should be noted, however, that the proof of uniqueness was obtained only for the values $\theta_0 \leq 54^\circ$. If this corresponds to the actual state of affairs and if the existence of the solution were established only for the values $\theta_0 \leq 54^\circ$ this would mean that the flow out of a symmetrical infinite vessel with straight walls is possible in the form of a steady continuous supersonic flow provided these walls include an angle not larger than 108°. The assumption is here made that for $2\theta_0 > 108^\circ$ supersonic flow without density jumps (shock waves) is impossible. It would be interesting to check this assumption experimentally.

With regard to the obtained solutions the curves of the velocity of sound start from the edges of the opening. It is to be noted that for $p = p_B$ there corresponds to the characteristics A'B' and AB one point of the plane (x, y), namely, the edge of the opening and this is also true for the case $p_1 < p_B$ with the corresponding characteristics A'E' and AE. This means that the flow in the neighborhood of the edges of the opening has the character of a Prandtl-Meyer flow (reference 5), that is, the character of the flow about a corner with expansion. The angle of inclination of the boundary of the jet as compared with the direction of the wall should be not less than $\theta_0/2$.

The flow within the vessel, since it is entirely determined by the solution ψ_1 of the problem of Tricomi, does not depend on the outside pressure p_1 provided $p_1 \leq p_B$. Hence the quantity of air

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*This proof must of course be completed with the proof that the Jacobian $D(x, y)/D(u, v)$ or the magnitude $(\partial\psi/\partial\sigma)^2 + K(\partial\psi/\partial\theta)^2$ for each of the solutions ψ_1, ψ_2, . . . has a constant sign. Otherwise the components would not be unique functions of the coordinates. In this case there would be expected the appearance of density jumps in the flow. It is not difficult to show to which types, according to Christianovich (reference 4) the flows considered belong. The flow ψ_1 in its supersonic part and also the flows ψ_2 and ψ_3 are mixed flows, the flow ψ_4 is a flow of rarefaction, the flows ψ_5 and ψ_6 are mixed flows, the flow ψ_7 is a flow of compression, etc.

per second likewise does not depend on p_1 as entirely corresponds to the well known experimental facts.

By the above indicated methods it will not, however, be possible to find a solution if

$$\left(\frac{2}{\kappa + 1}\right)^{\frac{\kappa}{\kappa - 1}} p_0 > p_1 > p_B \tag{26}$$

In this case the problem is reduced to a boundary problem which is a generalization of the problem of Tricomi. The solution is sought in the region OCD'B'A', where A'B' and CD' are arcs of the ellipsoid of Busemann and B'D' an arc of a circle corresponding to the given external pressure. The points O, A', C' are the same as in figure 4(a). The boundary conditions are the following:

$$\left.\begin{aligned} \psi &= -\frac{a}{2} \text{ on } OA'D'B' \\ \psi &= 0 \text{ on } OC \end{aligned}\right\} \tag{27}$$

The uniqueness of our solution has not yet been proven but is very probable. In the limit for $p = p^* = \left(\frac{2}{\kappa + 1}\right)^{\frac{\kappa}{\kappa - 1}} p_0$ the above boundary problem goes over into the Dirichlet problem and its solution into the solution of Chaplygin.

II. REDUCTION OF THE PROBLEM OF A SUPERSONIC FLOW ABOUT A WEDGE IN THE CASE OF THE FORMATION OF A SUBSONIC ZONE AHEAD OF THE WEDGE TO A BOUNDARY PROBLEM FOR THE EQUATION OF CHAPLYGIN IN AN INITIALLY KNOWN REGION OF THE VELOCITY PLANE THEOREM OF UNIQUENESS FOR THIS PROBLEM

In the case here considered the entropy behind the density jump is variable. In connection with this in the equation of Chaplygin for the flow there appears a part on the right side proportional to the derivative of the entropy with respect to the stream function (reference 5). In what follows we shall neglect this right part, as also in general the variability of the entropy. The flow then remains potential (reference 1) and the equations of Chaplygin remain in force:

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial\theta} &= -\frac{\partial\psi}{\partial\sigma}\\ \frac{\partial\varphi}{\partial\sigma} &= K\frac{\partial\psi}{\partial\theta}\end{aligned}\right\} \tag{1}$$

$$\frac{\partial^2\psi}{\partial\sigma^2} + K\frac{\partial^2\psi}{\partial\theta_2} = 0 \tag{2}$$

The problem of the flow about the wedge can now be reduced to a boundary problem in the region OABDE of the plane (u, v) (fig. 6). In this figure OA is a segment of the u axis, AB is the arc of the strophoid giving the velocity behind the wave front lying within the circle of the subsonic velocities. The equation of this strophoid (reference 5) is

$$\frac{v^2}{(V_1 - u)^2} = \frac{u - \dfrac{a^{*2}}{V_1}}{\dfrac{2}{\kappa + 1}V_1 + \dfrac{a^{*2}}{V_1} - u} \tag{3}$$

BD and ED are arcs of the characteristics (epicycloid), B and E lying on the circle of sound velocity, and OE is the radius making angle θ_0 where the latter is the angle between the sides of the wedge in the direction of the approaching flow. The boundary conditions are:

$$\psi = 0 \text{ on AOED} \tag{4}$$

$$\psi = \psi_B \text{ at point B} \tag{5}$$

where ψ_B is assumed given.

On the arc of the strophoid there must be satisfied such condition as would assure a continuous change in the stream function on passing through the wave front.

The corresponding transformation of the (u, v) plane into the (x, y) plane is shown in figure 6. As in the previous problem, to the characteristic ED corresponds a single point E-D in the (x, y) plane. At this point (corner at the base of the wedge) there arise flows of the Prandtl-Busemann type. The continuation of the flow beyond the Mach line OB is not of interest since it has no effect on the flow in front of this line (this continuation may be determined by the method of Prandtl-Busemann).

Under these conditions there will evidently be satisfied in the (u, v) plane those boundary conditions in the (x, y) plane, which are a consequence of the formulation of the problem of Chaplygin*. The value of ψ_B is proportional to the height of the wedge. It remains to render more precise the boundary conditions on the arc of the strophoid AB.

Let ρ_1 be the density in the undisturbed flow and λ the angle of inclination of the density jump at an arbitrary point (fig. 7). Let V_1 be the velocity of the undisturbed flow, ρ_0 the density at a stagnation point. We recall that $\rho_0 d\psi$ gives the difference in the discharge at two infinitely near points. Then along the discontinuity

$$\rho_0 d\psi = \rho_1 V_1 dy \tag{6}$$

On the other hand (reference 1)

$$dy = \frac{\partial y}{\partial \psi} d\psi + \frac{\partial y}{\partial \varphi} d\varphi = \frac{\rho_0}{\rho} \frac{\cos\theta}{V} d\psi + \frac{\sin\theta}{V}\left(\frac{\partial \varphi}{\partial \sigma} d\sigma + \frac{\partial \varphi}{\partial \theta} d\theta\right) =$$

$$= \frac{\rho_0}{\rho} \frac{\cos\theta}{V} d\psi + \frac{\sin\theta}{V}\left(K \frac{\partial \psi}{\partial \theta} d\sigma - \frac{\partial \psi}{\partial \sigma} d\theta\right) \tag{7}$$

From equations (6) and (7) we have

$$\left(\frac{1}{\rho_1 V_1} - \frac{\cos\theta}{\rho V}\right) d\psi = \frac{\sin\theta}{\rho_0 V}\left(K \frac{\partial \psi}{\partial \theta} d\sigma - \frac{\partial \psi}{\partial \sigma} d\theta\right) \tag{8}$$

Further (fig. 8)

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*Of the conditions which are satisfied on the wave front we have rejected one. This, however, is unavoidable since we have neglected the change of entropy.

$$V_1 \cos \lambda = V \cos (\lambda - \theta) = V_s \tag{9}$$

(that is, the tangential velocities do not change on passing through the discontinuity) and

$$\rho_1 V_1 \sin \lambda = \rho V \sin (\lambda - \theta) \tag{10}$$

(that is, the flow discharge does not change in passing through the discontinuity) (reference 1). Hence

$$\frac{1}{\rho_1 V_1} - \frac{\cos \theta}{\rho V} = \frac{1}{\rho_1 V_1}\left(1 - \frac{\sin (\lambda - \theta) \cos \theta}{\sin \lambda}\right) \tag{11}$$

and

$$\frac{V}{\sin \theta}\left(\frac{1}{\rho_1 V_1} - \frac{\cos \theta}{\rho V}\right) =$$

$$= \frac{1}{\rho_1} \frac{\cos \lambda}{\sin \theta \cos (\lambda - \theta)}\left(1 - \frac{\sin (\lambda - \theta) \cos \theta}{\sin \lambda}\right) = \frac{\operatorname{ctg} \lambda}{\rho_1} \tag{12}$$

Hence the boundary condition (8) becomes

$$K \frac{\partial \psi}{\partial \theta} d\sigma - \frac{\partial \psi}{\partial \sigma} d\theta = \frac{\rho_0}{\rho_1} \operatorname{ctg} \lambda \, d\psi \tag{13}$$

Since along the strophoid θ is a known function of σ the equation (13) gives a homogeneous linear relation between $\partial\psi/\partial\theta$ and $\partial\psi/\partial\sigma$.

We shall now show that the conditions (4), (5), and (13) determine the solution of equation (2) in the region OABDE uniquely, or in other words that the homogeneous conditions (4) and (13) determine the stream function except for a constant factor. For this purpose it is necessary and sufficient to show that the solution of equation (2) satisfying conditions (4), (13), and (5), $\psi(B) = 0$ must be identically equal to zero. To prove this it is sufficient to show that from the satisfying of the condition (4) along AOE, (13) along AB, and (5) at the point B there follows

$$\int_B^E \psi \frac{\partial \psi}{\partial \sigma} d\theta \leqslant 0 \tag{14}$$

where the integral is taken along the line $\sigma = 0$. For, in section 1 it has already been shown that due to the satisfying of the condition $\psi = 0$ along ED*

$$\int_B^E \psi \frac{\partial\psi}{\partial\sigma}\, d\theta \geqslant 0 \tag{15}$$

and that from (14) and (15) we have

$$\psi \equiv 0 \tag{16}$$

We shall prove inequality (14). We have:

$$0 = \iint\limits_{OAEB} \psi \left(\frac{\partial^2\psi}{\partial\sigma^2} + K\,\frac{\partial^2\psi}{\partial\theta^2}\right) d\theta\, d\sigma = -\iint \left[\left(\frac{\partial\psi}{\partial\sigma}\right)^2 + K\left(\frac{\partial\psi}{\partial\theta}\right)^2\right] d\theta\, d\sigma +$$

$$+ \oint \psi \left(K\,\frac{\partial\psi}{\partial\theta}\, d\sigma - \frac{\partial\psi}{\partial\sigma}\, d\theta\right) = -\iint \left[\left(\frac{\partial\psi}{\partial\sigma}\right)^2 + K\left(\frac{\partial\psi}{\partial\theta}\right)^2\right] d\theta\, d\sigma +$$

$$+ \frac{\rho_0}{\rho_1}\int\limits_{AB} \operatorname{ctg}\lambda \cdot \psi\, d\psi - \int_B^E \psi\,\frac{\partial\psi}{\partial\sigma}\, d\theta \tag{17}$$

Hence

$$\int_B^E \psi\,\frac{\partial\psi}{\partial\sigma}\, d\theta = -\iint\left[\left(\frac{\partial\psi}{\partial\sigma}\right)^2 + K\left(\frac{\partial\psi}{\partial\theta}\right)^2\right] d\theta\, d\sigma + \frac{\rho_0}{2\rho_1}\int_A^B \psi^2\,\frac{d\lambda}{\sin^2\lambda} \leqslant 0 \tag{18}$$

---

*It is here necessary of course, as in section 1, to assume that at the point D $M < 2$ (or that the central angle of the arc B is less than 54°.

which proves the uniqueness theorem*.

Since there was here assumed the validity of the transformations given it is necessary to supplement this proof of uniqueness by a proof of existence. To the problem of existence and the effective method of finding of a solution we hope to return later.

If the pressure behind the wedge is greater than the pressure at the point D (fig. 6) the region in the plane of the hodograph and the boundary conditions vary in the same manner as in the case of the flow out of a vessel (see remark to section I).

III. TWO LEMMAS TO THE THEORY OF THE EQUATION OF CHAPLYGIN

POSSIBILITY OF APPLICATION OF SERIES OF THE TYPE OF CHAPLYGIN TO THE PROBLEM OF A SUPERSONIC FLOW FROM A VESSEL

In this section we shall prove two lemmas to the equation of Chaplygin which we intend to use later for a proof of the existence of the solution of the problem of Tricomi. The first of these lemmas is an asymptotic formula for the logarithmic derivative of the function $z_\nu(\tau)$ of Chaplygin (see introduction, formula (11)) for $\tau = (2\beta + 1)^{-1}$ and large ν, namely

$$\frac{z'_\nu\left(\frac{1}{2\beta + 1}\right)}{z_\nu\left(\frac{1}{2\beta + 1}\right)} = C\nu^{2/3} + O\ (1) \tag{1}$$

where C is a constant independent of ν and the symbol $O\ (1)$ means a bounded magnitude.

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*The proof of uniqueness is applicable only if the central angle of the arc BE is less than 54°. In this connection there is also obtained the restriction for the angle θ_0 (fig. 6). The limiting angle θ_{0max} depends on the Mach number of the approaching stream. For $M = 1$, $\theta_{0max} = 54^\circ$, for $M = \infty$, $\theta_{0max} = 99^\circ$. The question as to whether this limiting angle has physical significance still remains open.

In preparing the paper "The Theory of the Laval Nozzle" for publication we were able to render this formula more accurate. We obtained an asymptotic formula for large ν:

$$\left.\frac{z'_\nu}{z_\nu}\right|_{\tau=\frac{1}{2p+1}} = C\nu^{2/3} + C_0 + C_1\nu^{-2/3} + \ldots + C_k\nu^{-\frac{2k}{3}} + 0\left(\nu^{-\frac{2k+2}{3}}\right)$$

In the case $\tau = \frac{1}{(2\beta + 1)}$ this formula is involved in an inequality proven by Chaplygin, namely, the inequality

$$\sqrt{1 - 2\beta s + 2\beta s^2 \sqrt[3]{\frac{\beta\ (1 + 2\beta)^2}{2\nu^2}}} > x_\nu > \sqrt{1 - 2\beta s + \frac{\beta s^2\ (1 + 2\beta)}{\nu + 1}} \tag{2}$$

where

$$x_\nu = 1 + \frac{\tau}{\nu}\frac{y'_\nu}{y_\nu} = \frac{\tau}{\nu}\frac{z'_\nu}{z_\nu} \tag{2a}$$

and

$$s = \frac{\tau}{1 - \tau} \tag{2b}$$

We now proceed to the proof of (1). By definition of the function $z_\nu(\tau)$, the function

$$\psi_\nu = z_\nu(\tau)\ \sin 2\nu\theta \tag{3}$$

satisfies the equation of Chaplygin. If we replace the variable τ by the variable σ (reference 1, section V, formula (91)) then from the equation

$$\frac{\partial^2\psi_\nu}{\partial\sigma^2} + K\frac{\partial^2\psi_\nu}{\partial\theta^2} = 0 \tag{4}$$

there follows

$$\zeta''_\nu(\sigma) - 4\nu^2 K\zeta_\nu(\sigma) = 0 \tag{5}$$

where

$$\zeta_\nu(\sigma) = z_\nu(\tau) \tag{5a}$$

We recall that

$$z_\nu(\tau) = \tau^\nu y_\nu(\tau) \tag{6}$$

where $y_\nu(\tau)$ is a solution of the hypergeometric equation

$$\tau(1 - \tau)\, y''_\nu + [2\nu + 1 + (\beta - 2\nu - 1)\tau]\, y'_\nu + \beta\nu(2\nu + 1)\, y_\nu = 0 \tag{7}$$

regular for $\tau = 0$.

Equation (7) has a second independent solution of the form

$$y^{(2)}{}_\nu(\tau) = \tau^{-2\nu} g_\nu(\tau) \tag{8}$$

where $g_\nu(\tau)$ is regular for $\tau = 0$. Therefore equation (5) has a second independent solution

$$\zeta^{(2)}{}_\nu(\sigma) = z^{(2)}{}_\nu(\tau) = \tau^{-\nu} g_\nu(\tau) \tag{9}$$

From the formula for the coefficient K (section 1, formula (4a)) it follows that near $\sigma = 0$

$$K = a\sigma + b\sigma^2 + \ldots \tag{10}$$

where

$$a = 2\left(\frac{2\beta + 1}{2\beta}\right)^{3\beta+1} \tag{10a}$$

Since for $\tau < (2\beta + 1)^{-1}$ (or $\sigma > 0$) K is bounded, it follows from equation (10) that

$$|K - a\sigma| < B\sigma^2 \text{ for } \sigma > 0 \tag{11}$$

In differential equation (5) we now replace the coefficient K by its approximate value equal to $a\sigma$. We obtain the equation

$$\bar{\zeta}''_\nu - 4\nu^2 a\sigma\bar{\zeta}_\nu = 0 \tag{12}$$

where $\bar{\zeta}_\nu(\sigma)$ is that solution of equation (12) which for $\sigma \to \infty$ $(\tau \to 0)$ approaches zero and which is equal to ζ_ν for $\sigma = 0$. This solution has the following form (reference 3, section III, formula (12)):

$$\bar{\zeta}_\nu(\sigma) = \lambda\left(\sqrt[3]{4\nu^2 a\sigma}\right)\frac{\zeta_\nu(0)}{\lambda(0)} \tag{13}$$

where

$$\lambda(\xi) = \int_0^\infty e^{-\frac{1}{2}\xi\rho-\frac{1}{3}\rho^3} \cos\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2}\xi\rho\right) d\rho \tag{14}$$

For what follows it is of importance that the function $\lambda(\xi)$ for any positive m satisfy the inequality

$$|\lambda(\xi)| < \frac{c}{\xi^m} \; (\xi > 0) \tag{15}$$

We now denote by $\delta\zeta_\nu$ the function

$$\delta\zeta_\nu = \zeta_\nu - \bar{\zeta}_\nu \tag{16}$$

This function satisfies the nonhomogeneous differential equation

$$(\delta\zeta_\nu)'' - 4\nu^2 K\delta\zeta_\nu = 4\nu^2 f(\sigma) \tag{17}$$

where

$$f(\sigma) = (a\sigma - K)\,\bar{\zeta}_\nu(\sigma) \tag{17a}$$

According to the general theory of homogeneous linear differential equations there follows from (17)

$$\delta\zeta_\nu(\sigma) = \frac{4\nu^2}{\Delta}\left\{\zeta^\nu(\sigma)\int_0^\sigma f(\sigma')\,\zeta^{(2)}{}_\nu(\sigma')\,d\sigma' + \right.$$

$$\left. + \zeta^{(2)}{}_\nu(\sigma)\int_\sigma^\infty f(\sigma')\,\zeta_\nu(\sigma')\,d\sigma'\right\} + C_1\zeta_\nu(\sigma) + C_2\zeta^{(2)}{}_\nu(\sigma) \tag{18}$$

where

$$\Delta = \begin{vmatrix} \zeta'_\nu & \zeta^{(2)\prime}{}_\nu \\ \zeta_\nu & \zeta^{(2)}{}_\nu \end{vmatrix}$$

It is easy to prove that

$$C_2 = 0 \tag{19}$$

In fact, for small τ (large σ)

$$\begin{gathered}\zeta_\nu(\sigma) = 0\ (\tau^{\nu}),\ \zeta^{(2)}{}_\nu = 0\ (\tau^{-\nu}),\ d\sigma = -\frac{d\tau}{2\tau}\,0\ (1)\\ f(\sigma) = 0\ (\sigma^2) = 0\ (\ln^2\tau) = 0\ (\tau^{-\epsilon})\end{gathered} \tag{20}$$

where ϵ is an arbitrarily small quantity. Thus

$$\int^{\sigma} f(\sigma')\ \zeta^{(2)}{}_\nu(\sigma')\ d\sigma' = \int_{\tau}^{(1+2\beta)-1} 0\ (\tau'^{-\nu-1-\epsilon})\ d\tau' \tag{21}$$

$$\int_{\sigma}^{\infty} f(\sigma')\ \zeta_\nu(\sigma')\ d\sigma' = \int_0^{\tau} 0\ (\tau'^{\nu-\epsilon-1})\ d\sigma' = 0\ (\tau^{\nu-\epsilon}) \tag{22}$$

From equations (16), (21), and (22) we have

$$\delta\zeta_\nu(\sigma) = 0\ (\tau^{-\epsilon}) + C_2\zeta^{(2)}{}_\nu(\sigma) \tag{23}$$

Hence if $C_2 \neq 0$ then

$$\lim_{\sigma\to\infty} \delta\zeta_\nu(\sigma) = \infty \tag{23a}$$

which contradicts the definition of this function

We shall now compute C_1. We have

$$\delta\zeta_\nu(0) = \frac{4\nu^2}{\Delta}\ \zeta^{(2)}{}_\nu(0)\int_0^{\infty} f(\sigma')\ \zeta_\nu(\sigma')\ d\sigma' + C_1\zeta_\nu(0) = 0 \tag{24}$$

whence we obtain for the constant:

$$C_1 = -\frac{4\nu^2}{\Delta}\,\frac{\zeta^{(2)}{}_\nu(0)}{\zeta_\nu(0)}\int_0^{\infty} f(\sigma')\ \zeta_\nu(\sigma')\ d\sigma' \tag{25}$$

and for the derivative $\delta\zeta'_\nu(0)$:

$$\delta\zeta'_\nu(0) = \frac{4\nu^2}{\Delta}\left\{\zeta^{(2)'}{}_\nu(0)\int_0^\infty f(\sigma')\,\zeta_\nu(\sigma')\,d\sigma'\right\} + C_1\zeta_\nu(0) =$$

$$= -\frac{4\nu^2}{\zeta_\nu(0)}\int_0^\infty f(\sigma')\,\zeta_\nu(\sigma')\,d\sigma' =$$

$$= -\frac{4\nu^2}{\lambda(0)}\int_0^\infty \lambda\left(\sqrt{4\nu^2 a\sigma'}\right)(a\sigma' - K)\,\zeta_\nu(\sigma')\,d\sigma' \tag{26}$$

so that

$$\frac{\zeta'_\nu(0)}{\zeta_\nu(0)} = \sqrt[3]{4a\nu^2}\,\frac{\lambda'(0)}{\lambda(0)} - \frac{4\nu^2}{\lambda(0)}\int_0^\infty \lambda\left(\sqrt{4a\nu^2\sigma'}\right)(a\sigma' - K)\,\frac{\zeta_\nu(\sigma')}{\zeta_\nu(0)}\,d\sigma' \tag{27}$$

However, according to Chaplygin (reference 1, vol. II, p. 30)

$$0 \leqslant \frac{\zeta_\nu(\sigma)}{\zeta_\nu(0)} \leqslant 1 \text{ for } 0 \leqslant \tau \leqslant (1 + 2\beta)^{-1} \tag{28}$$

$$(\text{or } 0 < \sigma)$$

Hence

$$\int_0^\infty \lambda\left(\sqrt{4a\nu^2\sigma'}\right)(a\sigma' - K)\,\frac{\zeta_\nu(\sigma')}{\zeta(0)}\,d\sigma' \leqslant \int_0^\infty \lambda(\xi)\,\frac{B\xi^2}{(4a\nu^2)^{2/3}}\,\frac{d\xi}{(4a\nu^2)^{1/3}} =$$

$$= 0\left(\frac{1}{\nu^2}\right) \tag{29}$$

Thus according to (27)

$$\frac{\zeta'_\nu(0)}{\zeta_\nu(0)} = \sqrt[3]{4a\nu^2}\,\frac{\lambda'(0)}{\lambda(0)} + 0\,(1) \tag{30}$$

which proves formula (1).

The computation of the magnitude $\lambda'(0)/\lambda(0)$ from formula (14) and the known properties of Γ functions gives

$$\frac{\lambda'(0)}{\lambda(0)} = -\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \cdot 3^{1/3} \tag{30a}$$

The second lemma is a consequence of the first and may be stated as follows: Let ψ be a bounded solution of equation (4) defined in the region

$$\sigma > 0, \quad 0 \leqslant \theta \leqslant \theta_0 \tag{31}$$

Let the limiting values for $\theta = 0$ and $\theta = \theta_0$ be

$$\psi(\sigma,\ 0) = \psi(\sigma,\ \theta_0) = 0 \tag{32}$$

Then there exists a kernel $K(\theta,\ \theta')$ not depending on ψ the properties of which are determined by the equation

$$K(\theta,\ \theta') = A\left(|\theta - \theta'|^{-1/3} - (\theta + \theta')^{-1/3} - (\theta + \theta' - 2\theta_0)^{-1/3}\right) + O(1) \tag{33}$$

which permits expressing the boundary values of ψ on the arc $\sigma = 0$ in terms of the boundary values of $\partial\psi/\partial\sigma$ for $\sigma = 0$ since

$$\psi(0,\ \theta) = \int_0^{\theta_0} K(\theta,\ \theta')\, \psi_\sigma(0,\ \theta')\, d\theta' \tag{34}$$

To satisfy equation (34) it is required merely that the square of the function $\psi_\sigma(0,\ \theta)$ be integrable.

We shall now prove this. We compute first the kernel $K(\theta,\ \theta')$ assuming that it exists. In the particular case

$$\psi = \psi_\nu = \frac{\zeta_\nu(\sigma)}{\zeta_\nu(0)} \sin 2\nu\theta \tag{35}$$

we have

$$\left.\frac{\partial\Psi_\nu}{\partial\sigma}\right|_{\sigma=0} = \frac{\zeta'_\nu(\sigma)}{\zeta_\nu(0)} \sin 2\nu\theta \tag{36}$$

We introduce the notation

$$\vartheta = \frac{\pi}{2\theta_0}\theta \tag{37}$$

$$\lambda_n = \frac{\zeta'_\nu(0)}{\zeta_\nu(0)} \tag{38}$$

Equation (36) may then be written in the form

$$\psi_\nu(0, \theta) = \frac{\psi_{\nu\sigma}(0, \theta)}{\lambda_n} = \int_0^{\theta_0} K(\theta, \theta')\,\psi_{\nu\sigma}(0, \theta')\,d\theta' \tag{39}$$

or

$$\frac{2\theta_0}{\pi}\int_0^{\pi/2} K(\theta, \theta') \sin 2n\vartheta' d\vartheta' = \frac{\sin 2n\vartheta'}{\lambda_n} \tag{40}$$

whence

$$K(\theta, \theta') = \frac{2}{\theta_0}\sum_{n=1}^{\infty}\frac{\sin 2n\vartheta \sin 2n\vartheta'}{\lambda_n} \tag{41}$$

It remains to investigate the convergence of this series. According to equation (30)

$$\lambda_n = \frac{\lambda'(0)}{\lambda(0)} = \sqrt{4a\nu^2}\left[1 + 0(n^{-2/3})\right] \tag{42}$$

$$\frac{1}{\lambda_n} = \frac{\lambda(0)}{\lambda'(0)\sqrt[3]{4a\nu^2}}\left[1 + 0(n^{-2/3})\right] \tag{42a}$$

whence

$$K(\theta,\ \theta') = \frac{2}{\theta_0}\frac{\lambda(0)}{\lambda'(0)\sqrt[3]{4a}}\sum_{n=1}^{\infty}\frac{\sin 2n\vartheta \sin 2n\vartheta'}{\nu^{2/3}} +$$

$$+ \sum_{n=1}^{\infty} \sin 2n\vartheta \sin 2n\vartheta'\ O(n^{-4/3}) \tag{43}$$

The second of the series on the right converges uniformly and the series

$$\sum_{n=1}^{\infty}\frac{\sin 2n\vartheta \sin 2n\vartheta'}{n^{2/3}} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\cos 2n(\vartheta - \vartheta')}{n^{2/3}} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{\cos 2n(\vartheta + \vartheta')}{n^{2/3}} \tag{44}$$

may be summed in explicit form.

It is sufficient to consider the series

$$\sum_{n=1}^{\infty}\frac{\cos n\varphi}{n^{2/3}}\ (\varphi \neq 0,\quad \varphi \neq 2\pi) \tag{45}$$

From the known formula for the Γ-function

$$\int_0^{\infty} e^{-t}\ t^{z-1}\ dt = \Gamma(z)$$

we have

$$\Gamma\left(\frac{2}{3}\right) n^{-2/3}\ e^{ni\varphi} = \int_0^{\infty} x^{-2/3}\ e^{-n(x-i\varphi)}\ dx \tag{46}$$

whence

$$\Gamma\left(\frac{2}{3}\right)\sum_{n=2}^{\infty} n^{-2/3}\ e^{ni\varphi} = \int_0^{\infty} x^{-1/3}\ \frac{e^{-x+i\varphi}}{1 - e^{-x+i\varphi}}\ dx \tag{47}$$

Formula (47) is obtained from (46) by summing the geometric series taking account of the fact that

$$\left|1 - e^{-x+i\varphi}\right| \geqslant \sin\varphi \text{ for } \begin{cases} 0 < \varphi \leqslant \frac{\pi}{2} \\ \frac{3\pi}{2} \leqslant \varphi < 2\pi \end{cases}$$

$$\left|1 - e^{-x+i\varphi}\right| < \sin\varphi \text{ for } \frac{\pi}{2} < \varphi < \frac{3\pi}{2}$$

and therefore

$$\int_0^\infty x^{-1/3} \frac{e^{N(-x+i\varphi)}}{1 - e^{-x+i\varphi}} dx \leqslant \frac{1}{|\sin\varphi|} \int_0^\infty x^{-1/3} e^{-Nx} dx =$$

$$= \frac{\Gamma\left(\frac{2}{3}\right)}{|\sin\varphi|} N^{-2/3} \longrightarrow 0 \text{ for } N \longrightarrow 0$$

or correspondingly

$$\int_0^\infty x^{-1/3} \frac{e^{N(-x+i\varphi)}}{1 - e^{-x+i\varphi}} dx \leqslant \Gamma\left(\frac{2}{3}\right) N^{-2/3}$$

Taking the real part of formula (47) we obtain the required sum of series (45):

$$\Gamma\left(\frac{2}{3}\right) \sum_{n=1}^{\infty} \frac{\cos n\varphi}{n^{2/3}} = \int_0^\infty x^{-1/3} \frac{e^{-x}\cos\varphi - e^{-2x}}{1 - 2e^{-x}\cos\varphi + e^{-2x}} dx \qquad (48)$$

We proceed to the investigation of the properties of the function (48) for $\varphi = 0$ and $\varphi = 2\pi$. It is sufficient of course to investigate the function (48) near $\varphi = 0$. We have:

$$\Gamma\left(\frac{2}{3}\right)\sum_{n=1}^{\infty}\frac{\cos n\varphi}{n^{2/3}} = \int_1^{\infty} x^{-1/3}\,\frac{e^x\cos\varphi - 1}{e^{2x} - 2e^x\cos\varphi + 1}\,dx +$$

$$+ \int_0^1 x^{-1/3}\,\frac{(1+x)\cos\varphi - 1}{(1+x)^2 - 2(1+x)\cos\varphi + 1}\,dx + O(1) =$$

$$= \int_0^1 x^{-1/3}\,\frac{(1+x)\cos\varphi - 1}{(1+x)^2 - 2(1+x)\cos\varphi + 1}\,dx + O(1) =$$

$$= \int_0^1 \frac{x^{2/3}\,dx}{\omega^2 + \varphi} + O(1) = \frac{3}{\sqrt[3]{\varphi}}\int_0^{\varphi^{-1/3}} \frac{z^4dz}{z^6 + 1} + O(1) \qquad (49)$$

The last integral is most simply computed with the aid of residues*.

Further,

$$\int_0^{\varphi^{-1/3}} \frac{z^4dz}{z^6 + 1} = \frac{1}{2}\int_0^{\varphi^{-1/3}} \frac{z^4dz}{z^6 + 1} =$$

$$\frac{1}{2}\left\{\frac{2\pi i}{6}\left[e^{-\pi i/6} + e^{-\pi i/2} + e^{-5\pi i/6}\right] + O(\varphi^{1/3})\right\} = \frac{\pi}{3} + O(\varphi^{1/3}) \qquad (50)$$

(The first terms come from the residues, and the term $O(\varphi^{1/3})$ from the integral over a semicircle of radius $\varphi^{-1/3}$)

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*For this remark which greatly simplifies the preliminary derivation the author is indebted to A. Nikolsky.

Thus we have finally

$$\Gamma\left(\frac{2}{3}\right)\sum_{n=1}^{\infty}\frac{\cos n\varphi}{n^{2/3}} = \frac{\pi}{\sqrt[3]{\varphi}} + \frac{\pi}{\sqrt[3]{2\pi - \varphi}} + O(1) \tag{51}$$

which gives for the kernal $K(\theta, \theta')$

$$K(\theta, \theta') = \frac{\lambda(0)}{\Gamma\left(\frac{2}{3}\right)\lambda'(0)\, a^{1/3}}\left\{|\theta - \theta'|^{-1/3} - (\theta + \theta')^{-1/3} - \right.$$
$$\left. - (2\theta_0 - \theta - \theta')^{-1/3}\right\} + O(1) \tag{52}$$

That the obtained kernel $K(\theta, \theta')$ actually expresses the boundary values of $\psi(\theta, 0)$ in terms of $\psi_\sigma(\theta, 0)$ is established first in the case where ψ_σ is expressed through a finite trigonometric series and in the second case by passing to the limit making use of the respresentation of ψ in the form of the Chaplygin series. Thus the second lemma has been proved. A similar lemma has been proven by Tricomi for the equation

$$y\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

It plays an essential part in reducing the problem considered to an integral equation of Fredholm of the second kind.

We now proceed to the question of the possibility of representing the solution of the problem of flow from a vessel with the aid of series of the type of Chaplygin. We recall the formulation of this problem. There is sought a bounded solution of equation (3) in a region of the (u, v) plane (see fig. 4) such that

$$\psi = 0 \text{ on OC}, \quad \psi = -\frac{Q}{2} \text{ on OA'B'} \tag{53}$$

We consider now a second solution ψ' of equation (3) defined by the equation

$$\psi' = \psi + \frac{Q}{2}\frac{\theta}{\theta_0} \tag{54}$$

This solution satisfies the following conditions of Tricomi:

$$\left.\begin{aligned} \psi' &= 0 \text{ on OC} \\ \psi' &= 0 \text{ on OA'} \\ \psi' &= \frac{Q}{2}\frac{\theta}{\theta_0} - \frac{Q}{2} \text{ on A'B'} \end{aligned}\right\} \tag{55}$$

We assume that this solution exists and that for $\sigma = 0$, $\partial\psi'/\partial\psi$ satisfies the condition proven by Tricomi in the case of equation (1)*

$$g(\theta) = \left.\frac{\partial\psi'}{\partial\sigma}\right|_{\sigma=0} = O(\theta^{-1/3}) \tag{56}$$

or in general that the square of the function $g(\theta)$ be integrable. Then

$$g(\theta) = \sum_{n=1}^{\infty} b_n \sin 2\nu\theta \tag{57}$$

where

$$\nu = \frac{\pi n}{2\theta_0} \tag{57a}$$

and $\sum_{n=1}^{\infty} b^2{}_n$ converges.

Then according to Chaplygin the solution ψ' in the sector of the circle OCA' will be

$$\psi = \sum_{n=1}^{\infty} \frac{b_n}{\lambda_n} \frac{\zeta_\nu(\sigma)}{\zeta_\nu(0)} \sin 2\nu\theta \tag{58}$$

where

$$\lambda_n = \frac{\zeta'_\nu(0)}{\zeta_\nu(0)} \tag{58a}$$

In particular on the arc CA'

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*In the case of analytic Laval nozzles this condition actually holds.

$$\Psi'(0,\ \theta) = f(\theta) = \sum_{n=1}^{\infty} \frac{b_n}{\lambda_n} \sin 2\nu\theta \tag{59}$$

As the results obtained by Chaplygin have shown, the convergence of the series (58) in the sector OCA' is assured. In the characteristic triangle CA'B' it is as yet impossible to say anything as regards the convergence of the series on the basis of these results. We have shown, however, in another paper (reference 7) that for continuous variation of the Cauchy data on the arc of the transition line ($\sigma = 0$) the solution of the equation of Tricomi in the corresponding characteristic triangle varies continuously. From this it follows that the problem of Tricomi stated by us is solved in the form of the series

$$\psi = -\frac{Q}{2}\frac{\theta}{\theta_0} + \sum_{n=1}^{\infty} a_n \frac{\zeta_\nu(\sigma)}{\zeta_\nu(0)} \sin 2\nu\theta \tag{60}$$

where the coefficients a_n are determined from the condition

$$\sum_{n=1}^{\infty} a_n \frac{\zeta_\nu[\sigma(\theta)]}{\zeta_\nu(0)} \sin 2\nu\theta = \frac{Q}{2}\left(\frac{\theta}{\theta_0} - 1\right) \text{ for } \frac{\theta_0}{2} < \theta < \theta_0 \tag{61}$$

$\sigma = \sigma(\theta)$ denoting the dependence of σ on θ along the arc of the epicycloid.

Thus out problem under the assumptions made has been reduced to the solution of an infinite system of ordinary linear equations. It may be attempted to solve this system approximately keeping only a finite number of terms in the infinite sum (equation (61)) and requiring only a correspondingly finite number of chosen values of θ.

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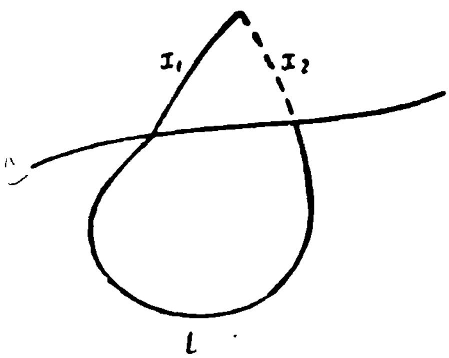


Fig. 1

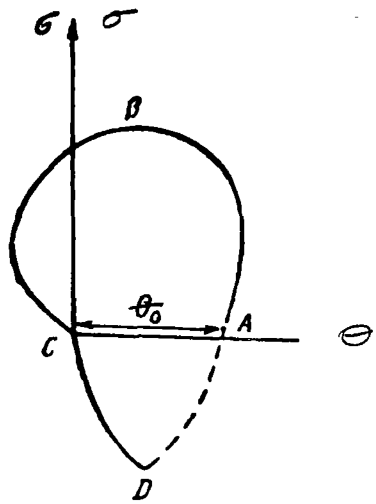


Fig. 2

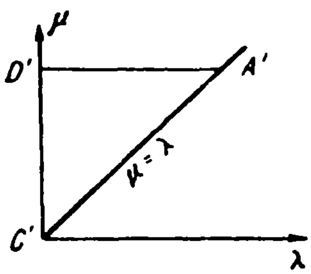
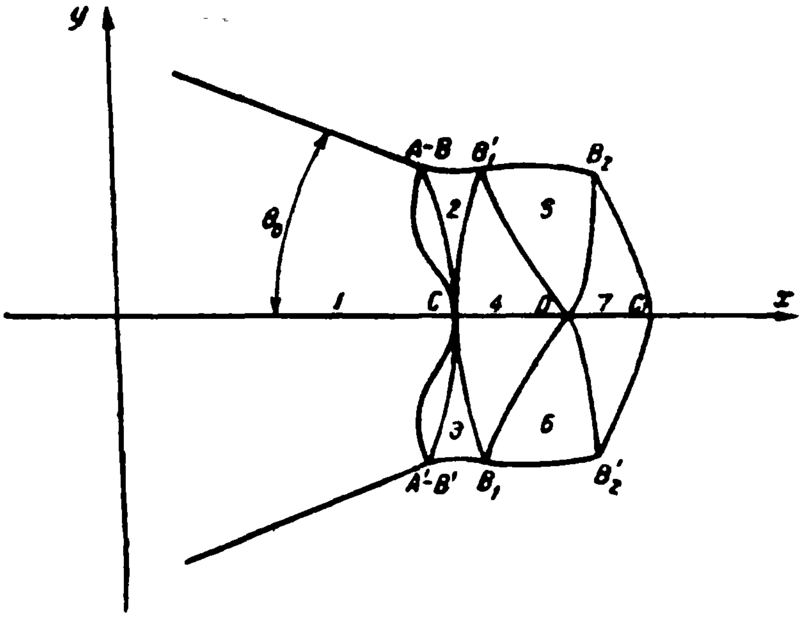


Fig. 3



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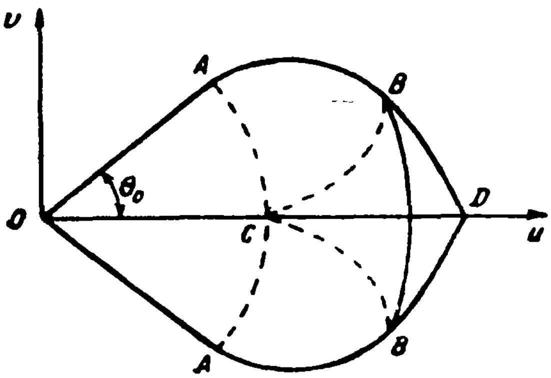
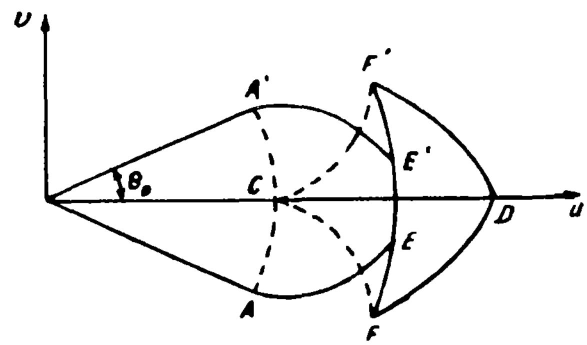


Fig. 4a



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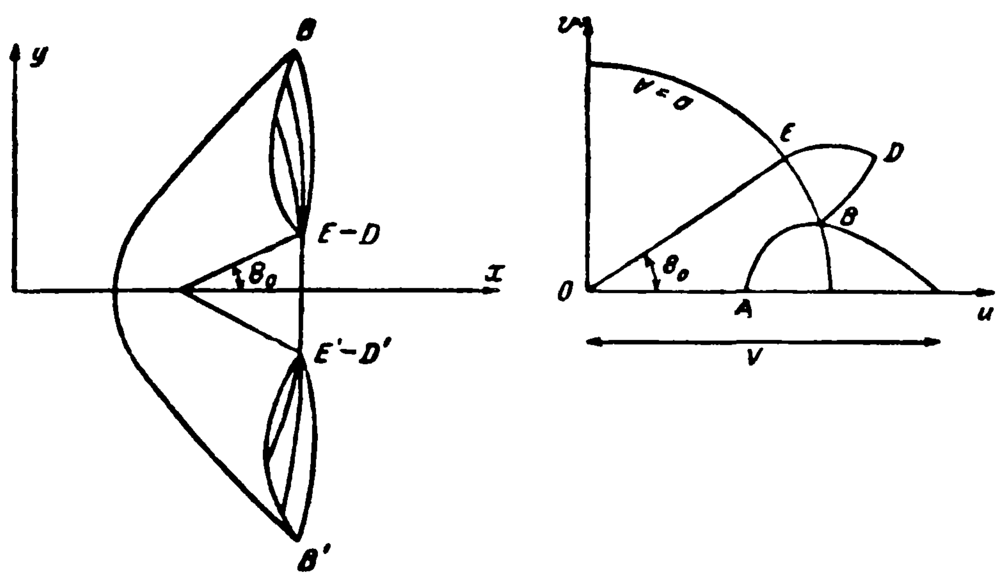

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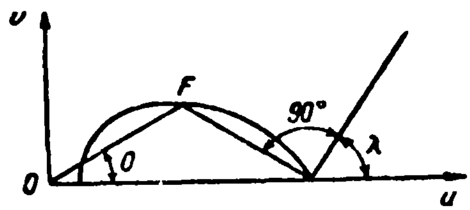

Fig. 7

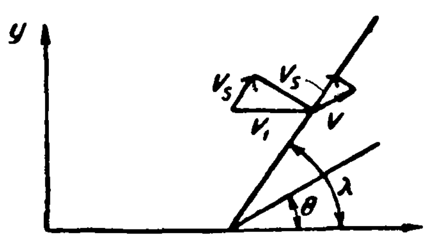

Fig. 8